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Modelling of rapid pressure–strain in Reynolds-stress closures

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The most general form for the rapid pressure–strain rate, within the context of *classical* Reynolds-stress transport (RST) closures for homogeneous flows, is derived, and truncated forms are obtained with the aid of rapid distortion theory. By a classical RST-closure we here denote a model with transport equations for the Reynolds stress tensor and the total dissipation rate. It is demonstrated that all earlier models for the rapid pressure–strain rate within the class of classical Reynolds-stress closures can be formulated as subsets of the general form derived here. Direct numerical simulations were used to show that the dependence on flow parameters, such as the turbulent Reynolds number, is small, allowing rapid distortion theory to be used for the determination of model parameters. It was shown that such a nonlinear description, of fourth order in the Reynolds-stress anisotropy tensor, is quite sufficient to very accurately model the rapid pressure–strain in all cases of irrotational mean flows, but also to get reasonable predictions in, for example, a rapid homogeneous shear flow. Also, the response of a sudden change in the orientation of the principal axes of a plane strain is investigated for the present model and models proposed in the literature. Inherent restrictions on the predictive capability of Reynolds-stress closures for rotational effects are identified.

1. Introduction

A basic aim and general trend in the development of turbulence models is to improve the range of validity of the models. The price to be paid in order to achieve the increased generality is a greater complexity of the models. However, mere algebraic complexity need not be an insurmountable obstacle today considering the availability and rapid rate of development of good hardware and software tools for computation.

In complex flows we typically encounter effects of strong streamline curvature or body forces caused by, for example, system rotation. Since the kinetic energy equation is unaffected by, for example, system rotation, the lowest level of single-point closure at which such effects enter explicitly is that in which transport equations are formulated for the individual Reynolds-stress components. In these, the Coriolis force gives rise to terms that will directly influence the intercomponent transfer. A nice demonstration of the capability of Reynolds-stress models to predict rotation effects is that of Launder, Tselepidakis & Younis (1987). They showed that even with relatively simple modelling of the terms involved, the main effects of system rotation on a plane turbulent channel flow could be predicted with reasonable accuracy. The tendency to develop a distinctly asymmetric velocity profile cannot be predicted with, for example, a standard k–ϵ model, but was here clearly shown to result from the inherent dynamics of the Reynolds stress transport equations.

The superiority of Reynolds-stress transport of (RST) closures compared to lower level models for the prediction of a rotating homogeneous shear flow was illustrated by Speziale, Gatski & Mac Giolla Mhuiris (1990), who also recognized, however, that there remain severe problems for cases with rotational mean flows. They particularly studied the case of rotating homogeneous shear flow. Other phenomena such as secondary flow in a non-circular-cross-section duct clearly need modelling at, at least, a level above the standard eddy viscosity level, since the flow in the cross-stream planes is driven by the anisotropic distribution of the Reynolds stresses.

The present study concerns the development of RST models, and in particular the role of intercomponent transfer caused by the part of the pressure–strain correlation associated with the mean velocity gradients. Near-wall pressure-reflection and strong inhomogeneity effects are outside the scope of the present work.

The basic foundations for the development of RST models in general and the treatment of pressure–strain-rate terms in particular were laid out by Chou (1945) and Rotta (1951). We will refer to closure schemes with transport equations for the velocity correlations, $\overline{u_i u_j}$, and the total dissipation rate, ϵ, as *classical* Reynolds-stress models. For more general background information the reader is referred to, for example, the review of Launder (1989). In the development of this type of closure scheme extensive use has been made of basic kinematical and other constraints that can be derived from the tensorial formulation of the equations (see e.g. Lumley 1978).

Instead of using $\overline{u_i u_j}$ as the quantity for which transport equations should be solved it is perhaps more convenient to formulate model equations for the kinetic energy, $k = \frac{1}{2}\overline{u_i u_i}$, and the Reynolds-stress anisotropy tensor

$$a_{ij} = \frac{\overline{u_i u_j}}{k} - \tfrac{2}{3}\delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Hence, in this context of classical Reynolds-stress closures all other turbulence quantities must be modelled as (algebraic) expressions of the mean velocity gradient tensor, k, ϵ and a_{ij} (the primary quantities). After having chosen the level of turbulence closure, one can regard the remaining task of modelling the individual terms as a question of how to, most effectively, use the information contained in these primary quantities.

In homogeneous turbulence there is no spatial redistribution of energy and the transport equations for the stress anisotropies can symbolically be written

$$\frac{\mathrm{d}a_{ij}}{\mathrm{d}t} = P_{ij}^{(a)} + \frac{1}{k}\Pi_{ij}^{(r)} + \frac{1}{k}\Pi_{ij}^{(s)} - \frac{\epsilon}{k}(e_{ij} - a_{ij}), \tag{1}$$

where the production term $P_{ij}^{(a)}$ is explicit in a_{ij} and hence needs no modelling. e_{ij} denotes the dissipation-rate anisotropy tensor

$$e_{ij} = \frac{\epsilon_{ij}}{\epsilon} - \tfrac{2}{3}\delta_{ij}$$

which traditionally has been taken to be zero, but some more ambitious modelling attempts have been made recently (see e.g. Hallbäck, Groth & Johansson 1990). The pressure–strain rate correlation term has in (1) been divided into a rapid or mean velocity–turbulence interaction term ($\Pi_{ij}^{(r)}$) and a slow or turbulence–turbulence interaction term ($\Pi_{ij}^{(s)}$). These represent intercomponent transfer and have zero trace.

The construction of more generally valid models for these terms is one of the key issues for improvement of Reynolds-stress closures.

The focus is here on energy redistribution terms in homogeneous turbulent flows. We will also restrict our attention to the rapid part (a recent investigation, heavily based on direct numerical simulations, of the slow part can be found in Hallbäck, Sjögren & Johansson 1993). For homogeneous flows the rapid pressure–strain rate can be written in terms of a fourth-rank tensor:

$$\Pi_{ij}^{(r)} = 4kU_{p,q}(M_{iqpj}+M_{jqpi}), \quad (2)$$

in which

$$M_{ijpq} = -\frac{1}{8\pi k}\int \frac{\partial^2 \overline{u_i u_j'}}{\partial r_p \partial r_q}\frac{\mathrm{d}\mathbf{x}'}{|\mathbf{r}|}, \quad \mathbf{x} = \mathbf{x}+\mathbf{r}, \quad (3)$$

and $U_{i,j}$ is the mean velocity gradient tensor. Expression (3) comes from the formal solution of the Poisson equation for the pressure field. This description is approximately valid also in moderately inhomogeneous flows as was illustrated by evaluation of numerical simulation data for turbulent channel flow (Bradshaw, Mansour & Piomelli 1987). For a more general case one may also regard (2) as the first term in a series expansion, where the next would involve mean flow curvature terms, etc.

In terms of spectral quantities we may write

$$M_{ijpq} = \frac{1}{2k}\int \frac{\kappa_p \kappa_q}{\kappa_m \kappa_m}\Phi_{ij}\,\mathrm{d}\boldsymbol{\kappa}, \quad (4)$$

where $\boldsymbol{\kappa}$ is the wavenumber vector and Φ_{ij} is the spectrum tensor. Crow (1968) considered the response of initially isotropic turbulence to a sudden rapid strain and was able to explicitly derive the initial magnitude of the rapid pressure–strain by use of (4).

It is immediately clear from (3) and (4) that the $\boldsymbol{M}$-tensor is symmetric with respect to the last two indices. It is also symmetric with respect to the first two, even if the spectrum tensor contains an antisymmetric part, since this helicity-related part always will vanish after integration over wavenumber space. Chou (1945) and Rotta (1951) did not construct any model for $\boldsymbol{M}$ explicitly, but were aware of the basic constraints of index symmetries and continuity. Rotta also noted that Green's theorem yields a further constraint, in that the Reynolds stresses are retrieved when the two last indices are contracted.

The above expressions show that $\boldsymbol{M}$ is not immediately affected by a sudden change in the mean strain field. This suggests that the modelled $\boldsymbol{M}$ should not explicitly depend on $U_{i,j}$. Furthermore, $\boldsymbol{M}$ is dimensionless and therefore cannot depend explicitly on k (or ϵ), except through a possible dependence on the turbulence Reynolds number. Altogether, this suggests that a natural approach to the modelling of the rapid pressure–strain is to express $\boldsymbol{M}$ in terms of the dimensionless Reynolds-stress anisotropy tensor. Actually, this is the only reasonable choice within the context of classical RST-modelling. Accordingly, practically all existing models can be seen as expansions of $\boldsymbol{M}$ to first or higher order in a_{ij}.

Reynolds (1987) (see also Lee & Reynolds 1985) analysed the modelling of $\Pi_{ij}^{(r)}$ by expanding $\boldsymbol{M}$ in terms of a_{ij}, and showed that the most general such expansion that satisfies correct index symmetry properties can be written with the aid of fifteen undetermined scalar functions. A complete ansatz for a linear model can be written in terms of five model parameters.

The complete linear $\Pi_{ij}^{(r)}$-model was first derived by Hanjalic & Launder (1972) (although not used there) and is described in detail in Launder, Reece & Rodi (1975).

Naot, Shavit & Wolfshtein (1973), who developed a model for the two-point velocity correlation obtained the same general form for the linear model through Taylor expansion of the correlation function. Launder *et al.* (1975) also demonstrated, through comparisons with various experimental data, the fairly wide range of applicability of such Reynolds-stress closures. Morris (1984) further studied the performance of linear models for inhomogeneous situations where (2 and 3) are not applicable. He derived a linear model by applying the symmetry and zero-trace conditions directly on the $\Pi_{ij}^{(r)}$ and tuned the resulting larger set of model parameters through comparisons with experiments (actually only homogeneous shear flow).

Lee (1989) studied in detail rapid distortion theory (RDT) solutions for axisymmetric turbulence undergoing contraction or expansion, with and without dilatation. He also compared different models for the rapid pressure–strain and constructed an extension of the linear model (for $\boldsymbol{M}$) that includes information about both the invariants of the anisotropy tensor and the magnitude of the strain rate. Considerable improvements in the predictions were obtained in this manner. The inclusion of strain-rate information in the model had earlier been attempted (see the discussion in Reynolds 1976 of an earlier model of Lumley), but was later abandoned because of the basic argument that the exact $\boldsymbol{M}$ is only indirectly dependent on the mean strain rate through the correlation function, and, hence, does not immediately respond a sudden change in the mean strain rate.

An interesting investigation of the performance of linear models was made by Shih, Reynolds & Mansour (1990) who presented a model for the spectrum tensor parametrized (to first order) in the Reynolds-stress anisotropy tensor. It bears a close resemblance to the parametrization of the spectrum tensor in spectral anisotropy measures described by Cambon, Jeandel & Mathieu (1981). Shih *et al.* showed that the resulting pressure–strain rate is equivalent to a linear model of the Launder *et al.* (1975) form. They found the range of validity of the spectrum model and thereby of linear $\Pi_{ij}^{(r)}$ models to be quite small in terms of the magnitude of the anisotropy. It is clear from their results that, in general, accurate predictions require a model that is nonlinear in the stresses, or equivalently, the anisotropy tensor. In fact, in 1972 Hanjalić & Launder had already proposed a nonlinear model, quadratic in the stresses, on the grounds of lack of agreement between linear models and experimental data.

It has been argued (see e.g. Reynolds 1976) that the model for the rapid pressure–strain should be linear since a field that is the sum of two uncorrelated fields should give a pressure strain that is the sum of the two individual pressure–strain rates (see (4)). This is in principle a sound argument, but one may readily show (see e.g. Lumley 1978) that linear models cannot satisfy the basic, and perhaps more vital, condition of ensuring realizable solutions under all flow conditions. Violation of this condition may under extreme conditions lead to prediction of negative energies.

The formulation of this realizability constraint is discussed in detail in Schumann (1977) and Lumley (1978). It is based on Schwartz' inequality and the simple fact that the velocity is real. A way to satisfy the realizability constraint is to require that the models should always ensure non-negative Reynolds stresses in a diagonalized system. Hence, we have an inequality condition for these quantities. Pope (1985) distinguishes between the concepts of weak and strong realizability. According to Pope the weak realizability principle states that if a physical quantity satisfies an inequality, the model should give values that satisfy the same inequality. The strong realizability principle furthermore states that the rate of change of a physical quantity in an extreme state is zero. This is a necessary condition for the possibility of extreme states to be accessible.

In the present context the condition of strong realizability would imply zero value for the modelled diagonal pressure–strain rate corresponding to a velocity component with vanishing energy.

Models for other terms in the stress transport equations have in some cases been proposed that satisfy the strong realizability constraint. An example is the nonlinear model for the dissipation-rate anisotropy tensor of Hallbäck *et al.* (1990).

The possibility of satisfying the realizability constraint and the improved prediction accuracy in general has led several investigators to develop nonlinear models (e.g. Shih & Lumley 1985 and Fu, Launder & Tselepidakis 1987). Shih, Mansour & Chen (1987) tested various models for the rapid pressure–strain against direct numerical simulation data for homogeneous turbulent flows and demonstrated clearly the improved accuracy in the predictions obtained with nonlinear models.

Speziale, Sarkar & Gatski (1991) analysed the modelling of the two parts of the pressure–strain, compared various existing models and proposed a new nonlinear model. They found an improved performance over linear models and recognized remaining difficulties in cases with rotational mean flows.

The numerical aspects of the implementation of nonlinear models, such as the possible influence on the convergence rate, have not yet been fully investigated.

The general increased predictive accuracy for nonlinear models can also be understood from the study of Shih *et al.* (1990) of a model for the spectrum tensor. A nonlinear $\Pi_{ij}^{(r)}$ model would correspond to a model spectrum with nonlinear parametrization in the anisotropies. The energy distribution in wavenumber space can thereby be considerably more complex than that of the model spectrum tensor studied by Shih *et al.* In general such a parametrization is an approximate description of the spectrum even if infinitely high-order nonlinearities are included in the parametrization. However, for rapidly irrotationally strained turbulence it appears possible to completely describe the spectrum tensor and its evolution through parametrization in a_{ij}. This can easily be verified for rapidly axisymmetrically strained initially isotropic turbulence. This also implies that classical RST-modelling can be expected to be adequate for such cases.

Models for the rapid pressure–strain normally contain a fair number of model parameters. Calibration of these has been done against experimental data and more recently against direct numerical simulation (DNS) results, and rapid distortion theory (RDT) has also proven to be a powerful tool in the determination of the model constants, although serious attention must, of course, be paid to the issue of possible dependence on the magnitude of the mean strain rate. One may note that in the rapid limit the slow pressure–strain and the dissipation become negligible and the only term left in (1) that needs modelling is the rapid pressure–strain rate.

Lee, Kim & Moin (1989) showed that RDT may indeed give an accurate description of the turbulence and its structure at realistic strain rates in a study, using RDT and direct numerical simulations, of turbulence subjected to a uniform mean shear. They used shear rates up to typical values that can be found in the near-wall region of turbulent channel flow, and found close similarities between results obtained with the two methods. RDT has also sometimes been used to estimate the energy distribution among the components in strongly sheared flows. For instance, initially axisymmetric turbulence in the situation of uniform rapid mean shear was analysed by Maxey (1982) with the main aim of obtaining asymptotic stress ratios. He also computed the pressure–strain-rate components for this case.

Lee (1990) used RDT to obtain the rapid pressure–strain rate in the case of irrotational mean flow. He derived a model expressed as an expansion (to fourth order) in the stress anisotropy tensor. This model is derived from a direct power series expansion of the RDT results without requiring the model to satisfy the realizability condition. Away from the two-component limit the model appears to give a good description of the rapid pressure–strain rate for irrotational mean flows.

Efforts to generalize the concept of RST-models have been discussed in the literature. Launder (1989) discussed the possibility of introducing transport equations for quantities other than the Reynolds stresses and the dissipation rate. He expresses some doubt though that the benefits from such an extension could outweigh the extra computational cost.

Mansour, Shih & Reynolds (1991) studied the effects of rapid rotation on initially anisotropic homogeneous turbulence with the aid of RDT and DNS, and discussed briefly some possibilities of extending the Reynolds-stress closure concept in order to improve the treatment of effects of rotation. Also, Cambon, Jacquin & Lubrano (1992) analysed a generalized approach with the aim of improving the description of effects of rotation. Some first steps were taken, and the great difficulty involved was recognized.

It is the aim of the present paper to investigate the limits of the accuracy obtainable for the modelling of the rapid pressure–strain-rate term within the context of Reynolds-stress closures. The ultimate form of the model is analysed and specific truncated models are evaluated with the aid of rapid distortion theory and direct numerical simulations. The derivation of the general form of the model is presented in §2, and comparisons with existing models are given in §3. Rapid distortion theory is used in §4 to determine model parameter values at different levels of truncation. Results over wide ranges of the total strain are given for four different homogeneous flows, viz. axisymmetric strain, plane strain, pure rapid rotation of initially anisotropic turbulence and homogeneous shear flow. Also, the response to a sudden change in the orientation of the principal axes of strain for a plane strain situation is investigated and compared with that of other models. The inherent limitations of Reynolds-stress closures in describing some of the aspects of the evolution of turbulence quantities in rotational mean flows are also illustrated. The results and conclusions are summarized in §5.

2. Derivation of the $\Pi_{ij}^{(r)}$ model

If we limit ourselves to the realm of classical RST closures the natural starting point for constructing a rapid pressure–strain-rate model is to assume that, apart from scalar parameters, the (dimensionless) **M**-tensor is a function of the Reynolds-stress anisotropy tensor alone. The most general ansatz for a model of **M** in terms of an expansion in a_{ij} may be written

$$\begin{aligned} M_{ijpq} = {} & A_1\,\delta_{ij}\,\delta_{pq} + A_2(\delta_{ip}\,\delta_{jq} + \delta_{iq}\,\delta_{jp}) \\ & + A_3\,\delta_{ij}\,a_{pq} + A_4\,a_{ij}\,\delta_{pq} + A_5(\delta_{ip}\,a_{jq} + \delta_{iq}\,a_{jp} + \delta_{jp}\,a_{iq} + \delta_{jq}\,a_{ip}) \\ & + A_6\,a_{ij}\,a_{pq} + A_7(a_{ip}\,a_{jq} + a_{iq}\,a_{jp}) + A_8\,a_{pk}\,a_{qk}\,\delta_{ij} + A_9\,a_{ik}\,a_{jk}\,\delta_{pq} \\ & + A_{10}(a_{ik}\,a_{pk}\,\delta_{jq} + a_{jk}\,a_{pk}\,\delta_{iq} + a_{ik}\,a_{qk}\,\delta_{jp} + a_{jk}\,a_{qk}\,\delta_{ip}) \\ & + A_{11}\,a_{ij}\,a_{pk}\,a_{kq} + A_{12}\,a_{pq}\,a_{ik}\,a_{kj} + A_{13}(a_{ik}\,a_{pk}\,a_{jq} \\ & + a_{jk}\,a_{pk}\,a_{iq} + a_{ik}\,a_{qk}\,a_{jp} + a_{jk}\,a_{qk}\,a_{ip}) \\ & + A_{14}\,a_{ik}\,a_{jk}\,a_{pl}\,a_{lq} + A_{15}\,a_{ik}\,a_{jl}(a_{kp}\,a_{lq} + a_{kq}\,a_{lp}). \end{aligned} \tag{5}$$

The scalar functions A_α, $\alpha = 1$–15 may depend on the invariants

$$\mathrm{II}_a = a_{ij}\,a_{ji}, \quad \mathrm{III}_a = a_{ij}\,a_{jk}\,a_{ki} \tag{6}$$

as well as the turbulence Reynolds number Re_T and possibly the strain-rate parameter S^*:

$$Re_T = \frac{4k^2}{\nu\epsilon}, \quad S^* = 2(\tfrac{1}{2}S_{ij}\,S_{ij})^{\frac{1}{2}}\frac{k}{\epsilon}, \tag{7}$$

where $S_{ij} = \frac{1}{2}(U_{i,j}+U_{j,i})$ is the strain-rate tensor. In the following analysis the mean velocity gradient tensor will be replaced by the sum of S_{ij} and the rotation tensor $\Omega_{ij} = \frac{1}{2}(U_{i,j}-U_{j,i})$. The possible dependence of A_α on the strain-rate parameter would contradict the previously discussed condition that the modelled $\boldsymbol{M}$ should not explicitly depend on the magnitude of the mean strain rate.

Expression (5) is complete, in that it contains all tensorially independent terms. The fact that fifth (and higher)-order terms do not give rise to further independent terms can easily be understood from the following reasoning: M_{ijpq} has four free indices and terms of fifth order in a_{ij} either contain one of the invariants (II_a or III_a) or a group like $a_{il}\,a_{lk}\,a_{kj}$. The latter can be rewritten with the aid of the Cayley–Hamilton theorem as

$$a_{il}\,a_{lk}\,a_{kj} = \tfrac{1}{3}\mathrm{III}_a\,\delta_{ij}+\tfrac{1}{2}\mathrm{II}_a\,a_{ij}. \tag{8}$$

Expression (5) was used by Reynolds (1987) and in some subsequent studies (see Lee 1990) to analyse the consequences for the modelling of the rapid pressure–strain rate. Insertion of the ansatz (5) in (2) would lead to an expression for the $\Pi_{ij}^{(r)}$ containing 14 scalar functions since A_1 and A_2 only appear in the combination A_1+A_2. However, some of the other terms can also be grouped together with the aid of the following three identities (readily verified by use of, for example, symbolic manipulation software):

$$a_{pk}(a_{ik}\,\delta_{jq}+a_{jk}\,\delta_{iq})\,S_{pq} \equiv (a_{ij}\,a_{pq}-a_{iq}\,a_{jp}+a_{pk}\,a_{kq}\,\delta_{ij}+\tfrac{1}{2}\mathrm{II}_a\,\delta_{ip}\,\delta_{jq})\,S_{pq}, \tag{9a}$$

$$a_{pk}(a_{ik}\,a_{jq}+a_{jk}\,a_{iq})\,S_{pq} \equiv (a_{ij}\,a_{pk}\,a_{kq}+a_{ik}\,a_{kj}\,a_{pq}-\tfrac{1}{3}\mathrm{III}_a\,\delta_{ip}\,\delta_{jq})\,S_{pq}, \tag{9b}$$

$$\begin{aligned} a_{pl}\,a_{qk}(a_{ik}\,a_{jl}+a_{jk}\,a_{il})\,S_{pq} \equiv\ & (2a_{ik}\,a_{kj}\,a_{pl}\,a_{ql}+\mathrm{II}_a(a_{ij}\,a_{pq}-a_{iq}\,a_{jp}) \\ & -\tfrac{2}{3}\mathrm{III}_a(a_{ip}\,\delta_{jq}+a_{jp}\,\delta_{iq}-\delta_{ij}\,a_{pq}))\,S_{pq}, \end{aligned} \tag{9c}$$

where a_{ij} and S_{ij} here actually may denote any traceless symmetric tensors of rank two. Relations (9*a*–*c*) can also be derived (after some algebra) from the generalized Cayley–Hamilton theorem for two-tensor products of extension three (see Spencer & Rivlin 1958). Hence, the number of scalar functions in the expression for $\Pi_{ij}^{(r)}$ is reduced to eleven with the aid of these identities. The continuity condition, which is equivalent to requiring $\Pi_{ij}^{(r)}$ to be traceless (see (2), (3)), is readily applied and further reduces the number of scalar functions to nine, and we get

$$\begin{aligned} \frac{1}{k}\Pi_{ij}^{(r)} = S_{pq}[&Q_1\,\delta_{ip}\,\delta_{jq}+Q_2(a_{ip}\,\delta_{jq}+a_{jp}\,\delta_{iq}-\tfrac{2}{3}a_{pq}\,\delta_{ij}) \\ &+Q_3\,a_{pq}\,a_{ij}+Q_4(a_{iq}\,a_{jp}-\tfrac{1}{3}a_{pk}\,a_{kq}\,\delta_{ij}) \\ &+Q_5\,a_{pl}\,a_{lq}\,a_{ij}+(Q_5\,a_{pq}+Q_6\,a_{pl}\,a_{lq})\,(a_{ik}\,a_{kj}-\tfrac{1}{3}\mathrm{II}_a\,\delta_{ij})] \\ &+\Omega_{pq}[Q_7(a_{ip}\,\delta_{jq}+a_{jp}\,\delta_{iq})+Q_8\,a_{pk}(a_{jk}\,\delta_{iq}+a_{ik}\,\delta_{jq})+Q_9\,a_{pk}(a_{jk}\,a_{iq}+a_{ik}\,a_{jq})]. \end{aligned} \tag{10}$$

The relations between the scalar functions $Q_\alpha = Q_\alpha(\mathrm{II}_a, \mathrm{III}_a, Re_T, S^*), \alpha = 1\text{–}9$, and the A_α are readily found by insertion of (5) in (2). The fact that the most general ansatz for $\Pi_{ij}^{(r)}$ can be written in terms of only nine scalar functions (or eleven before application of the continuity condition) has not been shown in earlier studies. Expression (10) constitutes the ultimate form for the model of $\Pi_{ij}^{(r)}$ (for homogeneous flows) within the context of Reynolds-stress closures, in which transport equations are used for $u_i u_j$ and ϵ (or equivalently a_{ij}, k, ϵ). Hence, all existing models based on the concept of expansion of $\boldsymbol{M}$ in terms of the stress anisotropy measures, can be expressed as subsets of (10).

2.1. *General conditions and model constraints*

Any reasonable model for $\boldsymbol{M}$ should at least conserve the index symmetry properties of the exact tensor and satisfy the continuity condition, the latter ensuring a traceless pressure–strain rate tensor. With the aid of definition (3) these conditions yield

$$M_{ijpq} = M_{ijqp}, \quad M_{ijpq} = M_{jipq}, \quad M_{ijiq} = 0. \tag{11}$$

The ansatz (5) satisfies the symmetry conditions and continuity is satisfied by the final form (10) for $\Pi_{ij}^{(r)}$. A further condition is obtained by the fact that $1/|\boldsymbol{r}|$ is the Green's function of the Laplace operator, so that the normalized Reynolds stress is retrieved when the last two indices of $\boldsymbol{M}$ are contracted. This so-called Green's condition,

$$M_{ijpp} = \tfrac{1}{2}a_{ij} + \tfrac{1}{3}\delta_{ij}, \tag{12}$$

is readily derived from definition (4) and gives rise, together with the continuity condition, to six equations for the A_α (in (5)). Inserting the results in the expressions for the Q_α, using the identities ($9a$–c) and observing that some of the A_α only appear in certain combinations, we may finally express all of these in terms of seven undetermined scalar functions, B_α, say:

$$Q_1 = \tfrac{4}{5} - \tfrac{2}{5}(4B_2 + 15B_3)\,\mathrm{II}_a - \tfrac{2}{5}B_5\,\mathrm{III}_a - \tfrac{1}{220}(19B_6 - 120B_7)\,\mathrm{II}_a^2, \tag{13a}$$

$$Q_2 = -12B_1 - \tfrac{1}{2}B_5\,\mathrm{II}_a - \tfrac{1}{2}(B_6 - 8B_7)\,\mathrm{III}_a, \tag{13b}$$

$$Q_3 = -8B_2 + 36B_3 + \tfrac{1}{22}(7B_6 - 72B_7)\,\mathrm{II}_a, \tag{13c}$$

$$Q_4 = 96B_2 - 36B_3 - \tfrac{1}{22}(7B_6 - 72B_7)\,\mathrm{II}_a, \tag{13d}$$

$$Q_5 = B_5, \tag{13e}$$

$$Q_6 = B_6, \tag{13f}$$

$$Q_7 = -\tfrac{4}{3} - \tfrac{28}{3}B_1 + \tfrac{1}{6}(2B_4 - B_5)\,\mathrm{II}_a - \tfrac{1}{18}(3B_6 - 56B_7)\,\mathrm{III}_a, \tag{13g}$$

$$Q_8 = -16B_2 + 28B_3 + \tfrac{1}{22}(3B_6 - 56B_7)\,\mathrm{II}_a, \tag{13h}$$

$$Q_9 = B_4. \tag{13i}$$

Hence, the complete $\Pi_{ij}^{(r)}$ can be expressed in terms of seven scalar functions (see (10)), and six of these appear in the part associated with the irrotational part of the flow (S_{ij}). Lee (1990) (see also Reynolds 1987) arrived at a form containing seven scalar functions associated with S_{ij}. The present results show that only six of these are independent.

It may here be interesting to note that the model ansatz for $\boldsymbol{M}$ that satisfies tensor symmetry conditions, continuity and the Green's condition also automatically reduces to the exact expression in the limit of isotropic turbulence:

$$M_{ijpq}^{(isotropic)} = \tfrac{2}{15}\delta_{ij}\,\delta_{pq} - \tfrac{1}{30}(\delta_{ip}\,\delta_{jq} + \delta_{jp}\,\delta_{iq}). \tag{14}$$

This also implies that the general form (10) for $\Pi_{ij}^{(r)}$, with the Q given by ($13a$–i), ensures a correct response to first order in time for the anisotropy measures in suddenly, rapidly distorted, initially isotropic turbulence, for arbitrary irrotational as well as rotational mean strain fields. Note that this is true regardless of the specifics of the scalar functions B_1–B_7.

2.1.1. *Realizability and truncation*

It follows from (2) and (3) that $\Pi_{\lambda\lambda}^{(r)}$ (no summation) vanishes in the limit of two-component turbulence where $u_\lambda \equiv 0$. The condition that this property should be conserved by the model is equivalent to the strong realizability principle (Pope 1985).

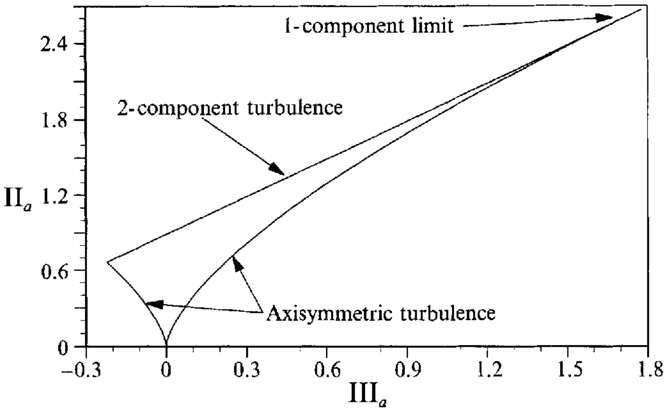

FIGURE 1. The a_{ij} invariant map.

This condition is described in more detail in the Appendix. In the case of modelling the rapid pressure–strain rate one may impose a somewhat more general condition,

$$M_{\lambda j p q} = 0 \tag{15}$$

which follows directly from definition (3) of $\boldsymbol{M}$.

The model ansatz (5) for $\boldsymbol{M}$ is an expansion in the Reynolds-stress anisotropy measures a_{ij}. It is, hence, natural to form the final model as a truncation of the expression at a chosen power of $\|a_{ij}\|$. To obtain such a consistent truncation we need to expand the scalar functions B_α, ($\alpha = 1$–7) in terms of the invariants II_a and III_a. A natural approach is to use a Taylor expansion in these quantities†. A consistent truncation at fourth order would include all tensorially independent terms in the model expression for $\boldsymbol{M}$, and contains the following set of scalar function expansions:

$$B_1 = \alpha_1 + \alpha_4 \mathrm{II}_a + \alpha_7 \mathrm{III}_a, \tag{16a}$$

$$B_2 = \alpha_2 + \alpha_8 \mathrm{II}_a, \quad B_3 = \alpha_3 + \alpha_9 \mathrm{II}_a, \tag{16b}$$

$$B_4 = \alpha_5, \quad B_5 = \alpha_6, \quad B_6 = \alpha_{10}, \quad B_7 = \alpha_{11}, \tag{16c}$$

where the eleven parameters may depend on Re_T and S^*, but not on the invariants of a_{ij}.

Truncations at third, second or first order are here obtained by including only $\alpha_1 - \alpha_6$, $\alpha_1 - \alpha_3$ and α_1, respectively. A fifth-order truncation would involve sixteen parameters, and so on.

The generalized realizability condition (15) can now readily be applied, and will reduce the number of undetermined parameters. The condition is to be applied in the two-component limit where all Reynolds stresses involving the empty component are zero. The invariants are in this limit related by

$$\mathrm{III}_a = \mathrm{II}_a - \tfrac{8}{9} \tag{17}$$

which, hence, can be represented by a straight line in the invariant map (see Lumley 1978). All turbulent states are, in the invariant map, located inside a region bounded by this line and the two curves representing axisymmetric turbulence (see figure 1). For axisymmetric turbulence the invariants are related by

$$\mathrm{II}_a^3 = 6\mathrm{III}_a^2. \tag{18}$$

† Other possibilities exist, and were investigated, but found not to add any greater generality to the final results. Other types of expansions will hence not be further discussed here.

For the fourth-order truncation six of the eleven parameters will be determined by the generalized realizability condition (15). We may, for instance, write

$$\alpha_4 = -\tfrac{3}{160}(3+60\alpha_1+48\alpha_2-40\alpha_3), \quad (19a)$$

$$\alpha_5 = -\tfrac{3}{2}-132\alpha_2+96\alpha_{10}, \quad (19b)$$

$$\alpha_6 = \tfrac{3}{24}+\tfrac{15}{4}(\alpha_2+\alpha_3), \quad (19c)$$

$$\alpha_7 = \tfrac{9}{8}\alpha_1-\tfrac{9}{4}(\alpha_2+\alpha_3)+\tfrac{1}{3}\alpha_{11}, \quad (19d)$$

$$\alpha_8 = -\tfrac{3}{88}(\tfrac{3}{8}+21\alpha_2+10\alpha_3+8\alpha_{10}), \quad (19e)$$

$$\alpha_9 = -\tfrac{9}{220}(\tfrac{3}{8}+21\alpha_2+10\alpha_3)+\tfrac{1}{11}\alpha_{11}. \quad (19f)$$

Five parameters remain, one of which (α_{11}), however, does not appear in the expressions for Q_α. Hence, four parameters, $\alpha_1, \alpha_2, \alpha_3, \alpha_{10}$, determine the complete fourth-order model for $\Pi_{ij}^{(r)}$. In the following we will denote these by γ_1, γ_2, γ_3 and γ_4, respectively. The latter has an essential influence for cases with rotational mean flow ($\Omega_{ij} \neq 0$). For irrotational cases it is of influence only when a_{ij} has non-zero off-diagonal elements in the coordinate system where the axes coincide with the principal axes of the mean strain-rate tensor.

The corresponding third-order model is obtained simply from the fourth-order model by setting

$$\gamma_2 = -\tfrac{3}{88}+\tfrac{21}{22}\gamma_1, \quad \gamma_3 = -\tfrac{3}{88}-\tfrac{5}{11}\gamma_1, \quad \gamma_4 = 0, \quad (20)$$

which, hence, has only one undetermined model parameter. The corresponding second-order model is found by setting

$$\gamma_1 = -\tfrac{1}{20}, \quad \gamma_2 = -\tfrac{1}{88}, \quad \gamma_3 = -\tfrac{3}{220}, \quad \gamma_4 = 0. \quad (21)$$

For the second-order model we note that all parameters are determined. This result has been obtained in several earlier studies (see e.g. Shih & Lumley 1985 and Reynolds 1987), where it also was shown that no linear model, based on this concept, can satisfy realizability.

The present methodology thus represents a new and quite straightforward (at least with the aid of symbolic manipulation software) technique to construct a model at a chosen level of truncation. The underlying assumption is that an expansion of the type (16*a*–*c*) has a reasonable behaviour within the complete invariant map. We will return later to the more far-reaching issue of whether the expansion may be said to, in some relevant sense, converge for increasing order of the truncation. The following presentation will be focused on the new fourth-order model and its relation to earlier models published in the literature.

2.2. *Dependence on Reynolds number and strain-rate parameter*

It is of particular importance for the generality of the final model that the dependence on various flow parameters either is small or can be described in a known manner. In the present work we have used direct numerical simulations to investigate the influence of Re_T and S^*.

2.2.1. *The simulation code*

The code used for the simulations is based on a pseudospectral method and can handle homogeneous flows with or without imposed mean strain fields. Mean strain or shear is handled by a distorting grid method (see Rogallo 1981). The code has been used in a large number of computations of homogeneous flows, in on-going work, for the general study of return-to-isotropy processes. It is a development of a simulation code for wall-bounded flows (see Lundbladh & Johansson 1991).

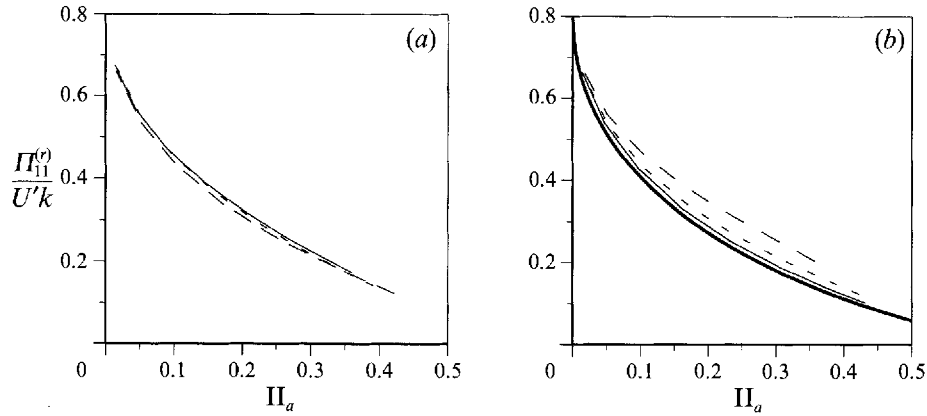

FIGURE 2. The normalized rapid pressure strain rate, $\Pi_{11}^{(r)}/kU'(U' = U_{1,1})$ as function of II_a in axisymmetric turbulence as obtained from direct numerical simulations. (*a*) Influence of $Re_T(S^* = 3)$: coarse-dashed, fine-dashed and solid curves represent $Re_T \approx 45$, 93 and 180, respectively. (*b*) Influence of $S^*(Re_T = 45)$: coarse-dashed, fine-dashed and solid curves represent $S^* \approx 1$, 3 and 9 respectively.

The computations presented below were carried out with up to 128^3 spectral modes. Care was also taken to investigate box size effects and to ensure that these were negligible.

Detailed evaluation of the various source terms in the Reynolds-stress-transport equations can easily be carried out when the complete velocity fields are available during the simulation. In the present investigation the rapid pressure–strain-rate term was determined from the DNS data by solving the Poisson equation for the so-called rapid part of the pressure field and thereafter forming the correlation with the fluctuating strain rate by averaging over the computational domain. Hence, ensemble averaging is replaced by volume averaging.

2.2.2. *Results for axisymmetric, strained turbulence*

Turbulence subjected to a homogeneous axisymmetric strain showed a practically negligible influence of the turbulent Reynolds number (see figure 2*a*) over the range investigated ($Re_T \approx 45$–180).

The influence of the strain-rate parameter S^* (for definition, see (7)) was found to be somewhat stronger, but still quite small over the range of values where the rapid part plays a dominant role over the slow pressure–strain rate (see figure 2*b*). For the cases shown in figure 2(*b*) this is actually only true for the highest strain rate. For rapidly axisymmetrically strained turbulence the kinetic energy increases essentially linearly with total strain. For the $S^* = 1$ case the strain rate is so slow that the kinetic energy decreases to half of its initial value during the simulation. The modelled normalized pressure–strain rate here deviates by about 20% from the RDT result. However, this will be of less importance since in the case of a low value of the mean strain rate the rapid pressure–strain term plays a minor role in the overall budget of the RST-equations. The small influence of the flow parameters in cases where the mean strain rate is large, and thereby $\Pi_{ij}^{(r)}$ plays a significant role in the balance equation, is encouraging for the possibility of constructing a model, where the model parameters, here denoted γ_1–γ_4, may be taken as numeric constants.

We may regard the simulation of three different mean strain rates in axisymmetric turbulence as a means of studying three different distributions of the energy in wavenumber space for a given degree of anisotropy. The general conclusion to be drawn from the simulation results is that there is a reasonable uniqueness in the relation between $\boldsymbol{M}$ and the anisotropy tensor a_{ij}. This is encouraging for the possibility of constructing a rapid pressure–strain-rate model.

Simulations at various strain rates were also carried out for the case corresponding to an axisymmetric expansion and a plane strain situation. The dependence on S^* was here found to be similar, and, hence, quite small for cases where the rapid effects dominate.

3. Existing $\Pi_{ij}^{(r)}$-models in Reynolds-stress closures

All existing models for the rapid pressure–strain rate that are based, explicitly or implicitly, on expansion of $\boldsymbol{M}$ in the Reynolds stresses, or equivalently their anisotropies, can be put in the form (10). Actually, this is essentially the only possibility, as long as we restrict ourselves to the class of *classical* Reynolds stress closures, where the transported quantities explicitly solved for are the Reynolds stresses and ϵ (or equivalently k, a_{ij} and ϵ).

The now classical linear $\Pi_{ij}^{(r)}$ model suggestions in the paper of Launder *et al.* (1975) have proven to be reasonably successful in flows with a moderate degree of anisotropy. In terms of the general form (10) their more general model (equation (10) in that paper) has non-zero values of the scalar functions Q_1, Q_2 and Q_7.

The fact that linear models cannot satisfy such basic properties as strong realizability has lead to efforts to generalize the model to include nonlinear terms. Nonlinear terms were actually already included in the paper of Hanjalić & Launder (1972). Shih & Lumley (1985) constructed a model that satisfies strong realizability and includes non-zero values for six of the nine scalar functions in the ultimate form (10). The nonlinear models of Fu *et al.* (1987) and Reynolds (1987) also satisfy strong realizability and can be seen as partial forms of (10). The model of Lee (1990) is derived by use of rapid distortion theory for irrotational strain fields. The description of these and the recent model of Speziale *et al.* (1991) in terms of the form (10) is summarized in table 1.

The performance of the various existing models for the rapid pressure–strain rate has been analysed in several recent papers. For instance, Shih & Lumley (1993) used DNS data for irrotational mean flows and homogeneous shear flow at different shear rates, and concluded that tensorially nonlinear models give predictions that are superior to those of linear models. They classified the model of Speciale *et al.* as quasi-linear, and pointed out that it does not satisfy the Green's condition. They suggested that the latter may be an important factor in explaining the relatively poor performance they found for that model.

The importance of satisfying basic conditions and constraints is obvious for the possibility of obtaining a model of reasonable generality. The strong realizability condition plays a special role in ensuring a sound behaviour near extreme states. In strongly strained or sheared flows the component(s) of vanishing energy will then be reasonably well predicted regardless of the details of model.

The performance of the present model(s) will be compared with that of existing models in the following sections.

Reference	Non-zero scalar functions	Satisfies SR
Launder *et al.* (1975)	Q_1–Q_2, Q_7	No
Shih & Lumley (1985)	Q_1–Q_4, Q_7–Q_8	Yes
Fu *et al.* (1987)	Q_1–Q_4, Q_7–Q_9	Yes
Reynolds (1987)	Q_1–Q_9†	Yes
Lee (1990)	Q_1–Q_6	No
Speziale *et al.* (1991)	Q_1–Q_3, Q_7	No

† Described in terms of 13 functions

TABLE 1. Existing rapid pressure–strain-rate models. SR denotes strong realizability

4. Determination of parameters for present model and comparisons with RDT and existing models

The simulation results indicate that the relation between $\boldsymbol{M}$ and the anisotropy tensor is practically independent of Reynolds number and little influenced by the variations in energy distributions in wavenumber space that may be caused by different mean strain rates. We may therefore in the following restrict ourselves to comparisons in the rapid limit where the results may be obtained by use of rapid distortion theory (RDT). In this limit we may write the transport (or evolution) equations for the stress anisotropies:

$$\frac{\mathrm{d}a_{ij}}{\mathrm{d}t} = P_{ij}^{(a)} + \frac{1}{k}\Pi_{ij}^{(r)}, \tag{22}$$

where the a_{ij}-production term can be expressed as

$$\begin{aligned} P_{ij}^{(a)} &= \frac{1}{k}(P_{ij} - \tfrac{1}{2}(a_{ij} + \tfrac{2}{3}\delta_{ij})\, P_{kk}) \\ &= S_{kl}(a_{ij}\, a_{kl} + \tfrac{2}{3}\delta_{ij}\, a_{kl} - \delta_{jl}\, a_{ik} - \delta_{il}\, a_{jk} - \tfrac{4}{3}\delta_{ik}\,\delta_{jl}) + \Omega_{kl}(\delta_{jl}\, a_{ik} + \delta_{il}\, a_{jk}). \end{aligned} \tag{23}$$

We note that $P_{ij}^{(a)}$ may be expressed as a quadratic form in the a_{ij} (cf. (10) for the pressure–strain rate term).

The general $\Pi_{ij}^{(r)}$ model form (10) inserted into (22) gives a correct behaviour of a_{ij} to first order in time (t) for initially isotropic, suddenly distorted turbulence, regardless of degree of truncation. For this type of flow situation it is natural to make comparisons between RDT-results and model results for increasing powers of t. We may, for small times, expand

$$a_{ij} = t a_{ij}^{(1)} + t^2 a_{ij}^{(2)} + \cdots \tag{24}$$

for the case of initially isotropic turbulence.

It may be shown that the model will give correct results to second order in time (i.e. $a_{ij}^{(2)}$ will also be given correctly by the model) by choosing

$$\gamma_1 = -\tfrac{1}{7} \tag{25}$$

for arbitrary irrotational or rotational mean strain fields, and regardless of the values of the other model parameters.

A model with truncation at third order in the *amplitude* of a_{ij} would be completely determined with the above choice for γ_1. Hence, a third-order model cannot be expected in general to give correct results for powers of t higher than two.

We note that a second-order $\Pi_{ij}^{(r)}$ model must have $\gamma_1 = -\frac{1}{20}$ in order to satisfy strong realizability, and can, hence, not give correct results to more than t^1 for arbitrary distortions.

At third order in time things get more complex. We will here first consider the case of arbitrary irrotational strain in some detail, exemplified by axisymmetric and plane strain. The situation with rotational mean flow will be illustrated by homogeneous shear and rapid pure rotation. Together these cases are used to determine the parameters, and to evaluate the capability of this type of modelling. A further test of the model is chosen as a case of a sudden change in the orientation of the principal axes of strain in a plane strain situation.

4.1. *Rapid irrotational strain – comparisons with RDT*

For the case of initially isotropic turbulence subjected to an arbitrary irrotational strain one can show that there is a unique relation between the anisotropy state, represented by a_{ij}, and the total strain, here measured in terms of $S_{ij}\,t$. It can be expressed as

$$a_{ij} = f_1 S_{ij}\, t + f_2 (S_{ik} S_{kj} - \tfrac{1}{3}\mathrm{II}_S\, \delta_{ij})\, t^2, \tag{26}$$

where we define the invariants of the mean strain-rate tensor as $\mathrm{II}_S = S_{ij} S_{ji}$ and $\mathrm{III}_S = S_{ij} S_{jk} S_{ki}$. Since only the total strain has influence we may write the scalar functions as $f_\alpha = f_\alpha(\mathrm{II}_S\, t^2, \mathrm{III}_S\, t^3)$ ($\alpha = 1, 2$). Note that (26) is an exact relation (not an expansion in time). The uniqueness of (26) rests on the fact that the spectrum tensor for this type of case with isotropic initial conditions is uniquely determined by the total applied strain alone (see e.g. Batchelor 1953 and Lee 1986). In a review of RDT and its applications Hunt (1978) discusses the applicability of RDT to Reynolds-stress closures, and rapid pressure–strain-rate modelling in particular. In agreement with the above formula he concludes that it is only for rapidly distorted irrotational flows that the turbulence statistics depend solely on the total strain (and not the entire straining history).

For small total strains one may express the scalar functions f_α in terms of an expansion in time:

$$f_\alpha = c_{\alpha 1} + c_{\alpha 2}\, \mathrm{II}_S\, t^2 + c_{\alpha 3}\, \mathrm{III}_S\, t^3 + c_{\alpha 4}\, \mathrm{II}_S^2\, t^4 + \cdots . \tag{27}$$

The coefficients of this expansion can be determined by inserting (27) into (26) and indentifying with the corresponding results of the RDT calculation. This yields†

$$(c_{11}, c_{21}, c_{12}, c_{22}, c_{13}, c_{23}, c_{14}, \cdots) = (-\tfrac{8}{15}, -\tfrac{8}{21}, \tfrac{184}{1575}, \tfrac{124}{1155}, \tfrac{248}{3465}, \tfrac{30512}{525525}, -\tfrac{112468}{3378375}, \cdots). \tag{28}$$

These RDT-coefficients were also derived by Lee (1990) in a similar manner. The inverse relation for the total strain expressed in terms of the anisotropy tensor can be written in a form exactly analogous to (26) and the scalar functions involved can readily be obtained from that relation.

In the case of initially isotropic turbulence a_{ij} will remain diagonal in the coordinate system aligned with the principal axes of the applied strain. The diagonal elements (the eigenvalues) of the anisotropy tensor may, with the aid of (26), (27), be written ($\alpha = 1$–3)

$$a_{\alpha\alpha} = c_{11}\, s_\alpha\, t + c_{21}(s_\alpha^2 - \tfrac{1}{3}\mathrm{II}_S)\, t^2 + c_{12}\, s_\alpha\, \mathrm{II}_S\, t^3 + [c_{13}\, s_\alpha\, \mathrm{III}_S + c_{22}(s_\alpha^2 - \tfrac{1}{3}\mathrm{II}_S)\,\mathrm{II}_S]\, t^4 + \cdots, \tag{29}$$

where s_α, $\alpha = 1$–3 are the eigenvalues of the mean strain tensor. In order to calibrate the model parameters we may now insert the model expression (10) for $\Pi_{ij}^{(r)}$ (with the Q_α determined from the Green's and realizability conditions) and $P_{ij}^{(a)}$ from (23) into (22), and expand a_{ij} in time. In the diagonalized reference system we obtain an

† All of this type of results were derived/checked by use of symbolic manipulation software to minimize the risk of algebra errors.

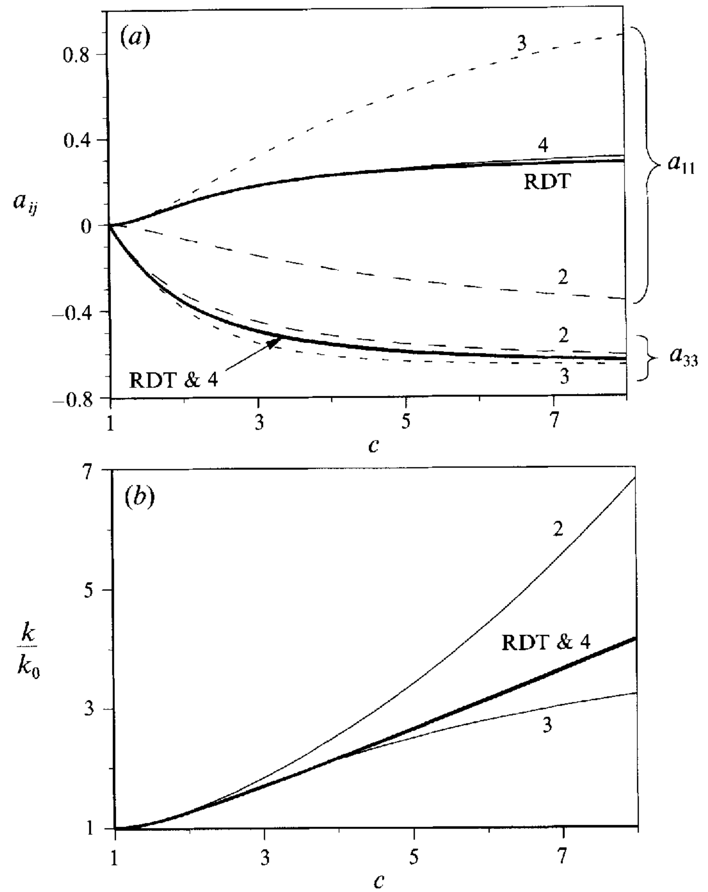

FIGURE 3. (*a*) The Reynolds-stress anisotropies and (*b*) kinetic energy as function of total reference strain in a (rapid) plane strain flow. The numbers 2–4 label the predictions with second-, third- and fourth-order models.

expression of the form (29) with coefficients dependent on the model parameters. Identification of powers of t with the corresponding RDT-expression yields, as expected, that the model always agrees to first order in time with RDT, and that second-order terms are identical if $\gamma_1 = -\frac{1}{7}$. Third-order terms agree with RDT if

$$\gamma_1 = -\tfrac{1}{7}, \quad \gamma_3 = \tfrac{9}{196} - \tfrac{16}{5}\gamma_2, \tag{30}$$

i.e. $a_{ij}^{(3)}$ in expansion (24) is correctly predicted by the model for arbitrary irrotational strain fields.

The fourth-order model will also give correct predictions to fourth order in time for the special case of plane strain (say, $s_1 = 0, s_3 = -s_2$) if

$$\gamma_1 = -\tfrac{1}{7}, \quad \gamma_2 = \tfrac{148179}{5027792} \approx 0.0295, \quad \gamma_3 = \tfrac{9}{196} - \tfrac{16}{5}\gamma_2 \approx -0.0484 \tag{31}$$

(actually a_{11} is correct to fifth order). Correct results to fourth order in time for arbitrary irrotational strains would require a fifth-order model. One should keep in mind that γ_4 does not influence this type of irrotational case (see comment in §2.1.1).

Model predictions for plane strain with these parameter values are compared with RDT results in figure 3. The results are shown as a function of total reference strain $c = \exp\left((\tfrac{1}{2}S_{ij}S_{ij})^{\frac{1}{2}}t\right)$, and show that the predictions converge rapidly towards the RDT results with increasing truncation order. One may note that the component with vanishing energy content is well predicted, not only by the fourth-order model, but also by the third- and second-order models. This can be ascribed to the built-in physics of the general form of the model and the fact that these models all satisfy the strong realizability condition. This property automatically ensures reasonable predictions near extreme states. Also, one should note that the calibration of the parameters is done for small times whereas the agreement between the predictions and RDT extends to quite large values of t (or c).

For the special case of irrotationally strained initially isotropic turbulence, the relation (26) and the corresponding inverse relation may be used together with (22) to derive an explicit expression for the rapid pressure–strain rate in the RDT limit. This expression can be written in the same form as (10) with $Q_5 = Q_6 = 0$. The explicit expressions for the Q_α are, however, not uniquely determined by this limit since the terms in (10) that are tensorially of second order are linearly dependent in the case where a_{ij} is diagonal in the reference system aligned with the principal axes of S_{ij}. Lee (1990) derived in a somewhat different manner an expression for $\Pi_{ij}^{(r)}$ from RDT in this type of situation (irrotational strain) and arrived at seven tensorially different terms (including up to fourth-order terms), all of which are not linearly dependent, though. Also Le Penven & Gence (1983) used RDT to derive an expression including up to second-order terms in the anisotropies.

In the case of rapidly strained axisymmetric turbulence $\Pi_{ij}^{(r)}$ can be uniquely determined from RDT. There is only one independent anisotropy measure $a_{11} = a$ in this situation ($s_2 = s_3 = -\frac{1}{2}s_1$), and the normalized pressure–strain rate can then be written

$$\frac{\Pi_{11}^{(r)}}{kU'} = \frac{4}{5} + \frac{12}{7}a + \frac{81}{98}a^2 + \frac{6075}{15092}a^3 + \frac{3692385}{5493488}a^4 + O(a^5), \tag{32}$$

where $U' = U_{1,1}$. The fourth-order model with the above choice (30) of the parameters is correct to second order in a, which is equivalent to third order in time for the anisotropies. The model predictions for the pressure–strain-rate term in this case are compared with RDT results and other models in figure 4.

The models that do not satisfy strong realizability, such as those of Launder *et al.* (1975) and Speziale *et al.* (1991), give rather poor predictions for situations with a strong degree of anisotropy. The model of Lee (1900) is derived directly from RDT results (for irrotational mean flow) and does not satisfy strong realizability. Yet, it gives accurate predictions except very close to the two-component limit. The model of Shih *et al.* (1987) (or Shih & Lumley 1985) satisfies realizability and may be said to be an extended second-order truncation model. It gives a good description of the axisymmetric case, but behaves essentially as the present second-order model for plane strain and homogeneous shear flow situations.

It can be concluded from figures 3 and 4 that a fourth-order model is sufficient to give accurate results (and satisfy realizability, etc.) for irrotational mean flows. It is also obvious from comparisons between figures 4 and 2(*b*) that the fourth-order model predictions are valid far outside the RDT regime, even down to non-dimensional mean strain rates of order unity.

4.2. *Rapid rotation – comparison with RDT*

The model parameter γ_4 has no influence on the previously described irrotational cases. A natural method of calibrating this parameter is by comparison with a pure rapid rotation. However, an initial isotropy is preserved by rapid rotation, and we must,

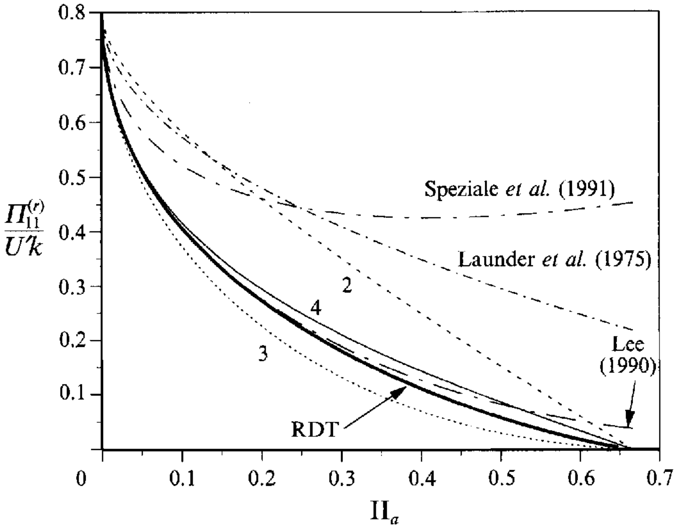

FIGURE 4. The predictions for the normalized rapid pressure strain rate, $\Pi_{11}^{(r)}/kU'$ as function of II_a in axisymmetric turbulence compared with the *exact* result obtained from RDT. The numbers 2–4 label the predictions with second-, third- and fourth-order models. Also included are results obtained with the models of Launder *et al.* (1975), Speziale *et al.* (1991) and Lee (1990). ($\gamma_1 = -1/20$ for second-order model predictions.)

hence, use an initially anisotropic situation for the present purpose. A relatively simple situation is that of an initially isotropic turbulence subjected to a plane strain ($s_1 = 0$, $s_2 = -s_3$). If such a field is subjected to a subsequent pure rapid rotation around the 1-axis, the anisotropy measures will exhibit damped oscillations. This can be found from a relatively straightforward rapid distortion analysis (see e.g. Cambon & Jacquin 1989 or Mansour *et al.* 1991). The asymptotic state will depend on the initial distribution of energy in wavenumber space, and the damping may be interpreted as a phase scrambling effect caused by internal waves. This phase scrambling is manifested in the way that each Fourier component will vary harmonically, but with a period that depends on the direction of the wavenumber vector. The phase relations will thereby be scrambled. It is natural to refer to the rotating coordinate system here (angular rotation rate ω). The frequency of oscillation of the anisotropy measures is close to 4ω (modified somewhat by damping effects).

Cambon *et al.* (1992) studied the effects of rotation by dividing the spectrum tensor, after neglecting the antisymmetric imaginary (helicity-related) part, into three parts:

$$\Phi_{ij}(\boldsymbol{\kappa}) = \underbrace{\frac{E(k)}{4\pi\kappa^2}\Delta_{ij}}_{\Phi_{ij}^{iso}} + \underbrace{\left(\tfrac{1}{2}\Phi_{kk}(\boldsymbol{\kappa}) - \frac{E(\kappa)}{4\pi\kappa^2}\right)\Delta_{ij}}_{\Phi_{ij}^{e}} + \underbrace{\Phi_{ij}(\boldsymbol{\kappa}) - \tfrac{1}{2}\Phi_{kk}(\boldsymbol{\kappa})\,\Delta_{ij}}_{\Phi_{ij}^{Z}},$$

where $\Delta_{ij} \equiv \delta_{ij} - \kappa_i\kappa_j/\kappa^2$, and $E(\kappa)$ is the three-dimensional energy spectrum. Only the first part has a non-zero trace after integration over spectral space, and thereby contributes to the kinetic energy. The two latter parts contribute to the anisotropy tensor $a_{ij} = a_{ij}^e + a_{ij}^Z$.

From the linearized dynamic equations, that are valid in the limit of rapid rotation, it follows that only Φ_{ij}^Z is affected by pure rotation. This implies that a_{ij}^e is constant under rapid rotations whereas a_{ij}^Z vanishes as a result of phase scrambling caused by internal waves.

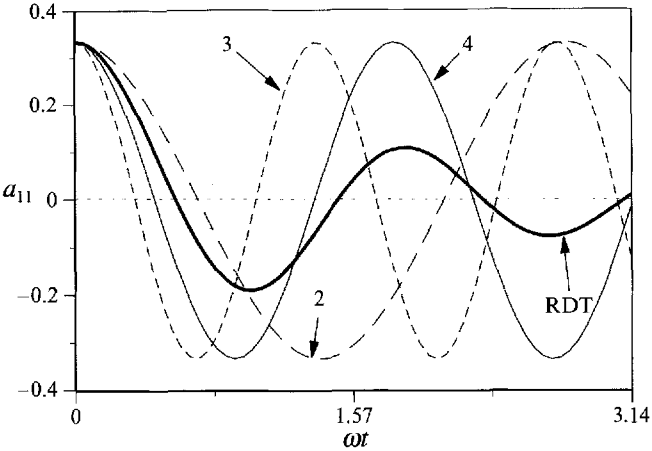

FIGURE 5. The anisotropy measure a_{11} as function of time in the rotating system for pure rapid rotation of an initial anisotropic state ($a_{11_0} = -a_{22_0} = \frac{1}{3}$). Also included is the RDT-result computed with an initial model spectrum according to Shih *et al.* (1990) with the free parameter chosen so that $a^e_{ij} = 0$ (note that the asymptotic state depends on the initial energy distribution in wavenumber space).

An interpretation of the contributions to the Reynolds-stress tensor from Φ^e_{ij} and Φ^Z_{ij} is that the former gives a measure of the 'dimensionality' of the energy distribution over lengthscales in the spatial directions and the latter is a measure of the distribution of energy among the velocity components ('componentiality'). One may note that there is a close relation to the structure anisotropy tensor y_{ij} used by Reynolds (1989) (see also Mansour *et al.* 1991): $a^e_{ij} = -y_{ij}$. This approach is valuable in the study of phenomena related to system rotation, but it does not solve the modelling problems concerning the rapid pressure–strain correlation. The modelling extension suggested by Cambon *et al.* is to include an *ad hoc* relaxation term that has no support in the original equations and definition of $\Pi^{(r)}_{ij}$.

For the class of classical Reynolds-stress closures we may note that a consequence of the assumption that $\boldsymbol{M}$ is expandable in a_{ij} alone is that all model predictions based on this concept will give undamped oscillations of the anisotropy measures. This can be seen from the fact that the general form (10) will imply conservation of the invariants II_a and III_a when $S_{ij} = 0$:

$$\frac{\mathrm{DII}_a}{\mathrm{D}t} = 2a_{ij}\left(P^{(a)}_{ij} + \frac{1}{k}\Pi^{(r)}_{ij}\right) \equiv 0, \tag{33a}$$

$$\frac{\mathrm{DIII}_a}{\mathrm{D}t} = 3a_{ik}\,a_{kj}\left(P^{(a)}_{ij} + \frac{1}{k}\Pi^{(r)}_{ij}\right) \equiv 0, \tag{33b}$$

independent of the specifics of the scalar functions Q_α. Thus, the most one can hope for within the concept of classical Reynolds-stress models is to predict a reasonable period of oscillation. With the present fourth-order model this is achieved with $\gamma_4 \approx 0.1$ for the case in figure 5. In fact, the optimal value may vary somewhat with the specifics of the initial conditions.

One may also note that a further consequence of this type of modelling is that initially axisymmetric turbulence subjected to rotation around the symmetry axis will be predicted to remain unaffected by rotation. In reality, oscillations of the anisotropy measures will occur here also, and will be damped by phase scrambling.

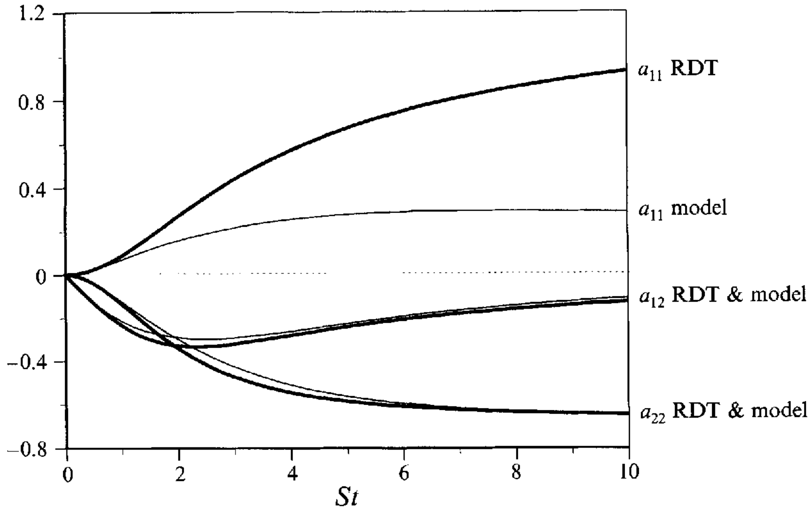

FIGURE 6. RDT and fourth-order model predictions of the Reynolds-stress anisotropies as function of non-dimensional time ($S \equiv U_{1,2}$) in a (rapid) homogeneous shear flow.

With these fundamental limitations in mind we may now consider all model parameters to be determined for the models with truncation level at fourth order or lower. The complete fourth-order model is now defined by the general form (10) an the definitions (13*a–i*), (16*a–c*), (19*a–f*) together with the above choice of model parameters, namely

$$\gamma_1 = -\tfrac{1}{7}, \quad \gamma_2 = 0.0295, \quad \gamma_3 = -0.0484, \quad \gamma_4 = 0.1. \tag{34}$$

4.3. *Tests of the model in two complex flow situations – comparison with existing models*

With the model parameters determined we may now test the predictive capability in two cases that involve considerable difficulties. The first is the idealized situation of a homogeneous shear flow, which despite its apparent simplicity offers considerable challenges to the modeller. It can be regarded as a superposition of a plane strain and a pure rotation. It is a situation where there is a misalignment between the principal axes of the anisotropy and strain-rate tensors. This misalignment cannot be predicted by simple models, for example of eddy-viscosity type, and offers together with the rotational part of the mean flow, interesting challenges. The second case is one where the orientation of the principal axes of the strain is suddenly changed in a plane strain situation, whereafter the adjustment of the Reynolds stresses to the new mean strain field is studied.

The homogeneous shear flow test case was studied under the conditions of rapid shear on a field of initially isotropic turbulence. Model predictions for the stress anisotropies are compared with RDT-results in figure 6. The corresponding comparison for the kinetic energy in figure 7 is complemented with predictions by other models published in the literature. The technique of obtaining RDT solutions for this case is described in, for example, Townsend (1970).

Considering the limitations in the capability to predict rotational effects, the results for the fourth-order model are still reasonably close to the RDT results. The models referred to earlier in this paper, included in figure 7, as well as the present second- and

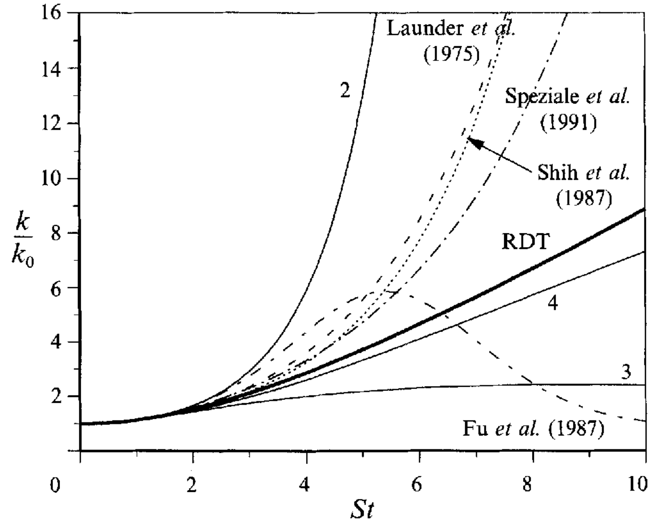

FIGURE 7. RDT and fourth-order model predictions of the kinetic energy as function of non-dimensional time in a (rapid) homogeneous shear flow. Also included are results obtained with the models of Launder *et al.* (1975), Fu *et al.* (1987), Speziale *et al.* (1991) and Shih *et al.* (1987).

third-order models give substantially worse results. For instance, the model of Shih *et al.* (1987) predicts an energy that is different by a factor of about 3 for $St = 8$.

Fifth- or higher-order truncations do not improve the situation significantly over that of the present fourth-order model. No truncation of this kind gives uniformly correct results in accordance with RDT for arbitrary mean velocity gradient fields (i.e. with non-zero Ω_{ij}).

The prediction of turbulent states for irrotational mean flows where the principal axes of the anisotropy and the mean strain rate tensors are misaligned offers new difficulties as compared with the previously described irrotational cases. The test case chosen here is a plane strain situation where the initial strain field is described by ($S_{11} = 0, S_{22} = -S_{33}$). After an initial period the strain field is rotated by 45° around the x_1-axis whereafter the adjustment of the Reynolds stresses is studied. The change in the orientation of the principal axes of the Reynolds stress (or anisotropy) tensor can be expressed as

$$\phi = \frac{1}{2}\arctan\frac{2a_{23}}{a_{22}-a_{33}}. \qquad (35)$$

One may note that, as a result of the two successive strains, fluid elements in fact will be rotated not just strained. The variation in the angle ϕ is shown in figure 8 where the RDT-results are compared with the present fourth-order, and other, models. The angle ϕ approaches 45° asymptotically. The complexity of the RDT-formulae for this case is such that their derivation calls for use of computer algebra.

The sudden change in slope at $c = 2$ in figure 8 also corresponds to an abrupt change in the slope of the kinetic energy and anisotropy measures. The model of Speziale *et al.* (1991) predicts well the variation of ϕ but does rather poorly for other quantities. One compact way of illustrating the variation of the turbulent state is to study the path in the anisotropy map, i.e. in the ($\mathrm{III}_a, \mathrm{II}_a$)-plane. Although both II_a and III_a undergo abrupt changes in time, the path in the ($\mathrm{III}_a, \mathrm{II}_a$)-plane is a smooth one. Figure 9 shows that the present fourth-order model gives excellent agreement with the path predicted by RDT, whereas the other models give results that are not even in qualitative agreement.

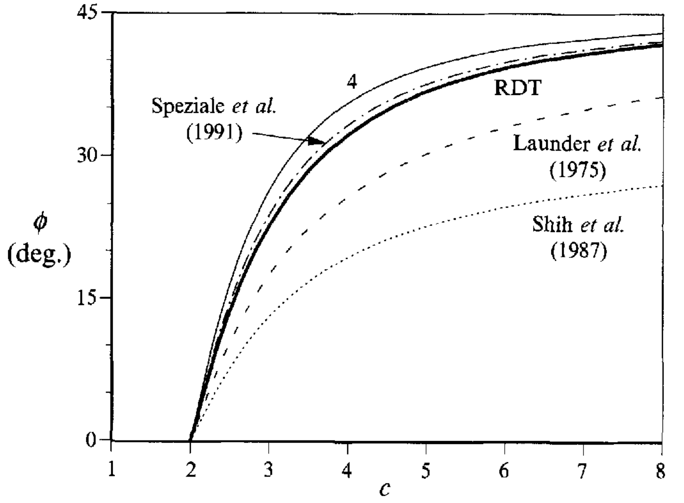

FIGURE 8. Predictions of the change in orientation of the principal axes of the Reynolds-stress tensor after a sudden 45° change in the orientation of the axes of applied plane strain. RDT and fourth-order model predictions are compared with the models of Launder *et al.* (1975), Speziale *et al.* (1991) and Shih *et al.* (1987).

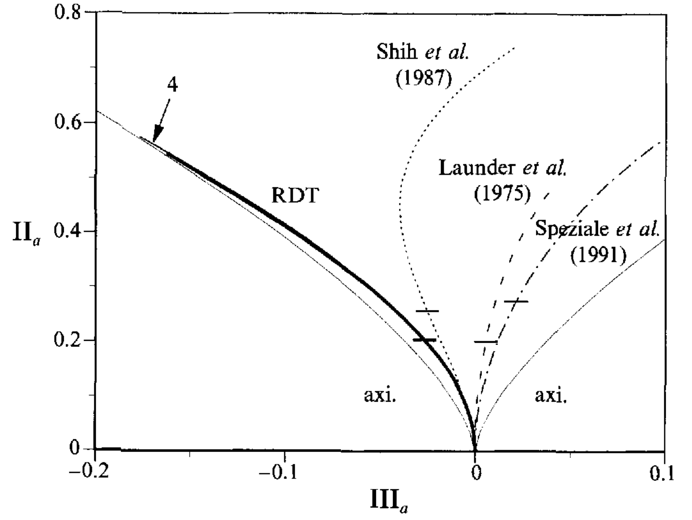

FIGURE 9. Predictions of the path in the anisotropy map after a sudden 45° change in the orientation of the axes of applied plane strain. RDT and fourth-order model predictions are compared with the models of Launder *et al.* (1975), Speziale *et al.* (1991) and Shih *et al.* (1987). The horizontal bars indicate where the orientation of the imposed strain was changed.

This is primarily due to misprediction of the third invariant, which is a quantity quite sensitive to details in the models. Note that here the model parameter γ_4 influences the result. The corresponding prediction in homogeneous shear flow is even more demanding and the situation is not as satisfactory as in figure 9, as is evidenced by the misprediction of a_{11} in figure 6.

5. Summary and conclusions

In classical Reynolds-stress closures, transport equations are formulated for the Reynolds stress tensor and the dissipation rate. Staying within this concept our only (reasonable) choice for the modelling of the rapid pressure–strain rate is in terms of an expansion of $\boldsymbol{M}$ in the anisotropy measures a_{ij}. It was shown here that the general form for $\Pi_{ij}^{(r)}$ based on such an expansion can be expressed in terms of seven scalar functions that in principle may depend on the invariants of a_{ij} and the other scalar flow parameters. The relation between $\boldsymbol{M}$ and a_{ij} was found, by use of direct numerical simulations, to be practically independent of Reynolds number and little influenced by the variations in energy distributions in wavenumber space that may be caused by different mean strain rates. This allows the use of rapid distortion theory for the determination of the model parameters as purely numeric constants. It also indicates that the assumptions underlying the one-point closure of the rapid pressure–strain rate are reasonably well satisfied.

Nonlinear models of second and higher order may be made to satisfy basic conditions, such as continuity, the Green's condition and strong realizability. By satisfying strong realizability one ensures prediction of non-negative energies, but also a generally sound behaviour near extreme states and a reasonably accurate prediction of energy-poor components, which is of particular value in strongly strained or sheared flows. The latter behaviour is, of course, not ensured by models that satisfy only the weak realizability condition.

At a truncation level of fourth order in the *amplitude* of a_{ij} four model parameters must be determined from comparisons with RDT. It was also shown from comparisons with DNS results that, although the model parameters are determined from comparisons with RDT at small times, the predictions are valid far outside this regime, down to quite moderate mean strain rates, and for large times. It was shown that a fourth-order model is quite sufficient to give very accurate predictions of irrotational mean flows, but also to get reasonable predictions of the kinetic energy development in a rapid homogeneous shear flow. Also the shear stress and the 'energy-poor' component are well predicted. There are limitations, though, inherent in the concept of classical Reynolds-stress closures for the predictive capability of the detailed features of the anisotropy state in this type of case (with $\Omega_{ij} \neq 0$).

As a severe test of the predictive capability of the present fourth-order model and other existing models, comparisons were made for a case where the orientation of the principal axes of the applied strain was given a sudden change. Of the models compared only the present one was able to predict the detailed changes in the turbulent state as measured by the Reynolds-stress anisotropy tensor.

Interesting possibilities have been suggested (Mansour *et al.* 1991 and Cambon *et al.* 1992) of generalizing the RST closure concept. The specific aim has been to improve the capability of predicting effects associated with rotational mean flow fields. Of course, the number of transport equations is also increased in these new approaches. Exciting new possibilities for future investigations can be seen here.

All in all, the present findings imply that it is not really possible to obtain a model with significantly better predictive capability or generality than that described by the present expression (10) along with a fourth-order truncation within the concept of classical Reynolds-stress closures.

This paper is dedicated to Mårten T. Landahl as a token of respect and gratitude by a scientific 'son' and 'grandson'. It is an extended version of a manuscript included in a Festschrift (*Progress in Fluid Mechanics* (ed. P. H. Alfredsson, F. H. Bark & A. V. Johansson)) published to celebrate the occasion of Mårten Landahl's sixty-fifth birthday. We would like to thank Anthony Burden and Erik Lindborg for many fruitful discussions and constructive critique of earlier versions of the paper. We also gratefully acknowledge financial support from NUTEK. Supercomputer time was provided by the National Supercomputer Center (NSC), Linköping, Sweden.

Appendix. Realizability

Since the eigenvalues of the Reynolds-stress tensor have the physical meaning of kinetic energies in three perpendicular directions they must always remain positive or, in an extreme state, be equal to zero. If we let $\sigma^{(\alpha)}$ and $v_i^{(\alpha)}$, with $\alpha = 1, 2, 3$, denote the eigenvalues and the eigenvectors, respectively, of the Reynolds-stress tensor R_{ij} their relationship may be expressed as

$$R_{ij}\, v_j^{(\alpha)} = \sigma^{(\alpha)} v_i^{(\alpha)},$$

from which we may obtain

$$\sigma^{(\alpha)} = v_i^{(\alpha)} R_{ij}\, v_j^{(\alpha)}$$

if we choose to work with normalized eigenvectors, i.e. $v_i^{(\alpha)} v_i^{(\alpha)} = 1$. In an extreme state where one eigenvalue vanishes, $\sigma^{(1)} = 0$ say, the first time derivative must be zero and the first non-zero derivative must be positive to ensure non-negative values of the eigenvalue itself. This is the strong realizability principle according to Pope (1985). We may first investigate to what extent the exact Reynolds-stress transport equation, as derived from the incompressible Navier–Stokes equations, satisfies realizability. The momentum equation for the fluctuating velocity may be written as

$$\dot{u}_i = f_i,$$

from which the Reynolds stress transport equation is obtained as

$$\dot{\overline{u_i u_j}} = \overline{u_i f_j} + \overline{u_j f_i},$$

where the force term f_i is given by

$$f_i = -U_k\, u_{i,k} - u_k\, U_{i,k} - (1/\rho) p_{,i} - (u_i\, u_k - \overline{u_i\, u_k})_{,k} + \nu u_{i,kk}.$$

The rate of change of an eigenvalue is given by

$$\dot{\sigma}^{(\alpha)} = \dot{v}_i^{(\alpha)} R_{ij}\, v_j^{(\alpha)} + v_i^{(\alpha)} R_{ij}\, \dot{v}_j^{(\alpha)} + v_i^{(\alpha)} \dot{R}_{ij}\, v_j^{(\alpha)},$$

where the first two terms are zero since they may be rewritten as

$$2\dot{v}_i^{(\alpha)} \sigma^{(\alpha)} v_i^{(\alpha)},$$

which evidently is zero since the eigenvectors are normalized. Thus, the rate of change of $\sigma^{(\alpha)}$ is obtained from the projection of the Reynolds-stress transport equation onto $v_i^{(\alpha)} v_j^{(\alpha)}$ (cf. Lumley 1978):

$$\dot{\sigma}^{(\alpha)} = v_i^{(\alpha)} (\overline{u_i f_j} + \overline{u_j f_i})\, v_j^{(\alpha)} = 2\overline{(v_i^{(\alpha)} u_i)\, (v_j^{(\alpha)} f_j)}.$$

In the two-component limit where $\sigma^{(1)} = 0$, say, the velocity field is plane and perpendicular to $v_i^{(1)} (\overline{(v_i^{(1)} u_i)^2} = 0)$. Hence, the right-hand side vanishes and $\dot{\sigma}^{(1)} = 0$ at the extreme state. In fact each term, originating from the corresponding term of the momentum equation, must vanish (cf. Schumann 1977). Concerning the modelling of the various terms this means that the strong realizability condition may be imposed on each of the following groups of terms:

$$
\begin{aligned}
&\text{advection:} && -U_k\,\overline{u_i u_{j,k}},\\
&\text{production:} && -\overline{u_i u_k}\,U_{j,k}-\overline{u_j u_k}\,U_{i,k},\\
&\text{triple correlation:} && -\overline{u_i u_j u_{k,k}},\\
&\text{pressure related:} && -\overline{u_i p_{,j}}-\overline{u_j p_{,i}} \equiv \overline{p(u_{i,j}+u_{j,i})}-\overline{pu_{i,j}}-\overline{pu_{j,i}},\\
&\text{viscosity related:} && \nu(\overline{u_i u_{j,kk}}+\overline{u_j u_{i,kk}}) \equiv -2\nu\overline{u_{i,k}u_{j,k}}+\nu\overline{u_i u_{j,kk}},
\end{aligned}
$$

where the traditional split of the pressure- and viscosity-related terms has been indicated.

In constructing turbulence models for the various terms, although rigorously not correct, as seen above, realizability is often imposed directly on the pressure–strain-rate tensor or the dissipation-rate tensor which represent only parts of the complete pressure or viscosity related terms. This approach can be justified by reasoning in the following way. The extreme states where the realizability condition becomes important are characterized by the velocity field being plane, i.e. $n_i u_i = 0$ for some vector n_i being constant over the complete ensemble of velocity fields at the specific spatial location at the specific time in question. We may ask ourselves in what types of situations such plane velocity fields may occur. In homogeneous turbulence a strong positive mean strain in the direction of n_i may cause a plane velocity field perpendicular to n_i. This is approximately the case when turbulence is strained by the passage through a wind tunnel contraction. In homogeneous turbulence the spatially redistributive terms are zero and we may therefore impose the realizability on the homogeneous parts of the source terms. Another case where we may encounter a plane velocity field is in the case of turbulence in the immediate vicinity of a solid wall. It is easily shown that each of the transport terms, i.e. the pressure velocity gradient and the triple velocity divergence terms and the viscous diffusion term, are identically zero at the wall in the equation for the stress normal to the wall. Thus, again, each of the terms in the traditional split satisfy realizability. It is difficult to find any other typical flow situation that may produce plane velocity fields. Thus, in practice, it is justified to apply the realizability condition on each of the terms in the right-hand side according to the traditional split, for instance on the homogeneous rapid pressure–strain-rate correlation, although it is in a mathematically strict sense an unnecessarily restrictive condition.

The requirement that the first non-zero time-derivative of $\sigma^{(1)}$ must be positive is normally not accounted for and is of course much harder to guarantee in an RST closure scheme.

Pope (1985) also discusses the principle of weak realizability, which states that in an extreme state $\dot{\sigma}^{(1)}$ must be non-negative. This also ensures non-negative eigenvalues, but may lead to situations where the extreme states are not accessible.

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